INVARIANTS OF CURVES AND SURFACES OF THE SECOND DEGREE BY THE GROUP OF MOTIONS AND THE GROUP OF SIMILITUDE.

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One of the most important and interesting problems of Lie's Theory of Continuous Groups is the determination of functions and systems of equations that are left invariant by a given group of transformations. The solution of the problem to determine when, by change of variables and parameters, a given group can be transformed into another given group, reduces itself to that of finding the invariant systems of equations by a certain group, and, on the other hand, many problems in geometry and the theory of differential equations depend upon the determination of invariant configurations.*

Given, in particular, a continuous group of r parameters and n variables that is generated by the r independent infinitesimal point-transformations:

$$X_1 f,\ X_2 f,\ \ldots,\ X_r f;$$

where

$$X_i f - \xi_{i1}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_1} + \xi_{i2}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_2} + \ldots + \xi_{in}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_n}.$$
$$i = 1, 2, \ldots, r.$$

The finite forms of the groups are

$$x_i' = x_i + \sum_1^r{}_k e_k X_k x_i + \frac{1}{1\,.\,2}\sum_1^r{}_k \Sigma_l e_k e_l X_k X_l x_i + \ldots, \quad i = 1, 2, \ldots, r$$

and any analytic function $f(x_1, \ldots, x_n)$ is transformed into the function

$$f'(x_1, \ldots, x_n) = f + \sum_1^r e_k X_k f + \frac{1}{1\,.\,2}\sum_1^r{}_k \Sigma_l e_k e_l X_k X_l f + \ldots, \quad i = 1, 2, \ldots, r$$

where, in both series, the e_k, e_l, ..., are constants.

Then if a function $I(x_1, \ldots, x_n)$ is to be an invariant function by the above r-parameter group there exists an identity of the form:

$$I(x_1, \ldots, x_n) + \sum_1^r{}_k e_k X_k I(x_1, \ldots, x_n) + \frac{1}{1\,.\,2}\sum_1^r{}_k \Sigma_l e_k e_l X_k X_l I(x_1, \ldots, x_n) + \ldots$$
$$= I(x_1, \ldots, x_n) \quad (1)$$

that must be true for all values of the constants $e_1, \ldots, e_r$.

*See Sophus Lie—Vorlesungen über continuierliche Gruppen, bearbeitet von Dr. Scheffers, Leipzig, 1893, pp. 404 et seq.

Hence

$$\sum_1^r e_k X_k I = 0\,,$$

or

$$X_k I \equiv 0\,, \qquad k = 1, \ldots, r \quad (2)$$

consequently,*

$$X_k X_l I \equiv 0\,, \ldots,$$

that is, I is an invariant function by the continuous group $X_1 f, X_2 f, \ldots, X_r f$ when the relations (2) are satisfied, and then only.

The equations (2) form a complete system of homogeneous linear partial differential equations of the first order, since

$$X_i(X_k f) - X_k(X_i f) \equiv \sum_1^r c_{iks} X_s f.\dagger \quad (3)$$

Accordingly the determination of the invariant function $I(x_1, \ldots, x_n)$ depends upon the integration of the complete system

$$X_1 I = 0\,,\ X_2 I = 0\,,\ \ldots,\ X_r I = 0\,. \quad (4)$$

This system contains r equations and n variables, and consequently has at least $n - r$ solutions. The problem of discovering the invariant functions by a group of infinitesimal point-transformations is thus referred by Lie to the integration of a complete system of homogeneous linear partial differential equations of the first order. Hence the number of invariant functions of n variables by a group of r parameters is not less than $n - r$. In particular cases the number of invariant functions may be greater than $n - r$ in consequence of the vanishing of certain determinants of the matrix formed by the coefficients of the $\frac{\partial f}{\partial x_j}$ in the complete system (4). The maximum number is obviously $n - 1$.

The solution above is independent of any special property of the infinitesimal point-transformations by which the continuous group considered is generated. It holds true, then, for a group of projective infinitesimal transformations. By an infinitesimal point-transformation, point is transformed into point, curve into curve, and surface into surface; the projective trans-

* This conclusion is also clear geometrically when we interpret $I(x_1, \ldots, x_n) =$ *constant* as a surface in space of n-dimensions. The equations (2) then express that the surfaces $I(x_1, \ldots, x_n) = a$, are generated by the path-curves of every infinitesimal point transformation $\Sigma e_k X_k f$ of the given group. Hence when $X_k \equiv 0$, then also $X_k X_l \equiv 0$, etc.

† In general the equations (2) form a complete system by virtue of the conditions (3) both when the c_{iks} are constants and when some or all are functions of the variables. But since, in the case in hand, the $X_i f$ form a continuous group, the first principles of Lie's theory demand that here the c_{iks} be constants.

formation possesses these properties together with the additional characteristic that the degree of the locus is also preserved by the transformation, that is, a surface of the mth degree goes over into a surface of the mth degree, and a curve of the nth degree into a curve of the nth degree.

Hence when the infinitesimal point-transformations $X_k f$ are projective the above method may be applied to determine the invariants by any projective group of 1° a surface of the mth degree ; 2° a curve of the nth degree ; 3° a system of points, of general or restricted position ; 4° any system consisting of a finite number of points, curves, and surfaces.*

The values of the increments δx_i given to the variables x_i by the transformations are expressed explicitly in the transformations themselves. The increments $\delta_r a_j$ of the parameters a_j which enter into the equations of condition, namely, the equations of the figures that make up the geometric configuration whose invariants are to be determined, can be found by means of these equations. The $\delta_r a_j$ are obviously functions of the a_j alone.

It is proposed here to find the invariants† of a conic 1° by the group of motions, 2° by the group of similitude, in the plane ; and to solve the corresponding problems in ordinary space, namely those of determining the invariants of a conicoid 1° by the group of motions, 2° by the group of similitude, in space of three dimensions.

I.

Considering the problem in the plane, let

$$U = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{5}$$

be the equation of the conic. Let

$$I(a, h, b, g, f, c)$$

be an invariant function of the coefficients of $U = 0$ by any projective group of infinitesimal transformations in the xy-plane. Then, since U is homogeneous in the parameters, I must be homogeneous and of the zero order, hence by Euler's theorem, we have

$$a\frac{\partial I}{\partial a} + h\frac{\partial I}{\partial h} + b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} + c\frac{\partial I}{\partial c} = 0 . \tag{6}$$

* Here the particular projective group considered is subject to but one limitation, namely : that it be a projective transformation in a space of dimensions not lower than the space of highest dimensions represented by any individual of the configuration whose invariant by the group is sought. For example, the method would not be applied to finding the invariants by the projective group of the plane of a system one of whose numbers is a sphere.

† The term "invariant" is here used in the sense of an "absolute invariant;" the determinant of substitution of Cayley's theory of invariants is here equal to unity. Lie has drawn a sharp distinction between invariant expressions and invariant equations. Vid. loc. cit. pp. 716 et seq.

Also since I is invariant, the variation of I must be zero, hence the equation

$$\delta I \equiv \frac{\partial I}{\partial a}\,\delta a + \frac{\partial I}{\partial h}\,\delta h + \frac{\partial I}{\partial b}\,\delta b + \frac{\partial I}{\partial g}\,\delta g + \frac{\partial I}{\partial f}\,\delta f + \frac{\partial I}{\partial c}\,\delta c = 0\,. \qquad (7)$$

The group of motions in the plane is a subgroup of the general projective group and contains the three infinitesimal point-transformations :

$$\boxed{p \quad q \quad yp - xq}^{\,*} \qquad (8)$$

The group of similitude in the plane, also projective, is a four-parameter group and consists of the four infinitesimal point-transformations :

$$\boxed{p \quad q \quad yp - xq \quad xp - yq} \qquad (9)$$

where $p \equiv \dfrac{\partial f}{\partial x}$, $q \equiv \dfrac{\partial f}{\partial y}$.

Operating on $U = 0$ with the several infinitesimal transformations of these groups, we obtain the following results :

$$\begin{aligned} pU\dagger &\equiv 2ax + 2hy + 2g + x^2\delta_1 a + 2xy\delta_1 h + y^2\delta_1 b + 2x\delta_1 g \\ &\qquad + 2y\delta_1 f + \delta_1 c = 0\,, & (10)\\ qU &\equiv 2hx + 2by + 2f + x^2\delta_2 a + 2xy\delta_2 h + y^2\delta_2 h + 2x\delta_2 g \\ &\qquad + 2y\delta_2 f + \delta_2 c = 0\,, & (11)\\ (yp - xq)\,U &\equiv 2axy + 2hy^2 + 2gy - 2hx^2 - 2bxy - 2fx + x^2\delta_3 a + 2xy\delta_3 h \\ &\qquad + y^2\delta_3 b + 2x\delta_3 g + 2y\delta_3 f + \delta_3 c = 0\,, & (12)\\ (xp + yq)\,U &\equiv 2ax^2 + 2hxy + 2gx + 2hxy + 2by^2 + 2fy + x^2\delta_4 a + 2xy\delta_4 h \\ &\qquad + y^2\delta_4 b + 2x\delta_4 g + 2y\delta_4 f + \delta_4 c = 0\,. & (13)\end{aligned}$$

These equations exist by virtue of the equation $U = 0$; hence comparing them successively with $U = 0$, we have the following systems, respectively :

$$\frac{\delta_1 a}{a} = \frac{\delta_1 h}{h} = \frac{\delta_1 b}{b} = \frac{\delta_1 g + a}{g} = \frac{\delta_1 f + h}{f} = \frac{\delta_1 c + 2g}{c} = \rho_1\,; \qquad (14)$$

$$\frac{\delta_2 a}{a} = \frac{\delta_2 h}{h} = \frac{\delta_2 b}{b} = \frac{\delta_2 g + h}{g} = \frac{\delta_2 f + b}{f} = \frac{\delta_2 c + 2f}{c} = \rho_2\,; \qquad (15)$$

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* Lie uses this notation to indicate that the infinitesimal transformations enclosed form a continuous group.

† The notation pU, qU, . . . , is used for convenience only, in order to enable the reader to identify the results of the operations. It is not strictly exact, since here the $X_i f$ include also arbitrary operators giving rise to the terms in $\delta_i x_j$.

$$\frac{\delta_3 a - 2h}{a} = \frac{\delta_3 h + a - b}{h} = \frac{\delta_3 b + 2h}{b} = \frac{\delta_3 g - f}{g} = \frac{\delta_3 f + g}{f} = \frac{\delta_3 c}{c} = \rho_3; \quad (16)$$

$$\frac{\delta_4 a + 2h}{a} = \frac{\delta_4 h + 2h}{h} = \frac{\delta_4 b + 2b}{b} = \frac{\delta_4 g + g}{g} = \frac{\delta_4 f + f}{f} = \frac{\delta_4 c}{c} = \rho_4 . \quad (17)$$

The solution of these systems of equations for the variations of the parameters gives:

$$\delta_1 a = \rho_1 a \quad , \delta_1 h = \rho_1 h \quad , \delta_1 b = \rho_1 b \quad , \delta_1 g = \rho_1 g - a,$$
$$\delta_1 f = \rho_1 f - h, \; \delta_1 c = \rho_1 c - 2g ; \quad (18)$$

$$\delta_2 a = \rho_2 a \quad , \delta_2 h = \rho_2 h \quad , \delta_2 b = \rho_2 b \quad , \delta_2 g = \rho_2 g - h,$$
$$\delta_2 f = \rho_2 f - b, \; \delta_2 c = \rho_2 c - 2f ; \quad (19)$$

$$\delta_3 a = \rho_3 a + 2h, \; \delta_3 h = \rho_3 h - (a - b), \; \delta_3 b = \rho_3 b - 2h, \; \delta_3 g = \rho_3 g + f,$$
$$\delta_3 f = \rho_3 f - g, \; \delta_3 c = \rho_3 c ; \quad (20)$$

$$\delta_4 a = \rho_4 a - 2a, \; \delta_4 h = \rho_4 h - 2h \quad , \delta_4 b = \rho_4 b - 2b, \; \delta_4 g = \rho_4 g - g,$$
$$\delta_4 f = \rho_4 f - f, \; \delta_4 c = \rho_4 c .^* \quad (21)$$

These systems of values must satisfy the equation (7), respectively. They contain the four indeterminate multipliers $\rho_1, \rho_2, \rho_3, \rho_4$, but these will be found to cause no inconvenience, since by virtue of the partial differential equation (6) they disappear from the results of the substitutions in equation (7) of the above values of the variations of the parameters; in the results ρ appears only as the coefficient of the left hand member of equation (6). Making the substitutions successively in the above order we have respectively:

$$a \frac{\partial I}{\partial g} + h \frac{\partial I}{\partial f} + 2g \frac{\partial I}{\partial c} = 0 , \quad (22)$$

$$h \frac{\partial I}{\partial g} + b \frac{\partial I}{\partial f} + 2f \frac{\partial I}{\partial c} = 0 , \quad (23)$$

$$2h \left[\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b} \right] - (a - b) \frac{\partial I}{\partial h} + f \frac{\partial I}{\partial g} - g \frac{\partial I}{\partial f} = 0 , \quad (24)$$

$$2a \frac{\partial I}{\partial a} + 2h \frac{\partial I}{\partial h} + 2b \frac{\partial I}{\partial b} + g \frac{\partial I}{\partial g} + f \frac{\partial I}{\partial f} = 0 . \quad (25)$$

* These values of the variations of the parameters might also have been found in this manner: Form the total variation of $U = 0$, viz. $\delta_r U = 0$ where δ_r operates on both variables and parameters, substitute in the resulting equations the values of $\delta_r x$ and $\delta_r y$ as expressed by the infinitesimal transformations considered; compare the new equations respectively with $U = 0$, and we obtain as before the system (14) . . . (17).

1. Then to determine the invariants of the conic $U = 0$ by the group of motions we have the following complete system of homogeneous linear partial differential equations of the first order :

$$
M_2 \begin{cases}
a \dfrac{\partial I}{\partial g} + h \dfrac{\partial I}{\partial f} + 2g \dfrac{\partial I}{\partial c} = 0\,, & (22) \\[2ex]
h \dfrac{\partial I}{\partial g} + b \dfrac{\partial I}{\partial f} + 2f \dfrac{\partial I}{\partial c} = 0\,, & (23) \\[2ex]
2h \dfrac{\partial I}{\partial a} - (a - b) \dfrac{\partial I}{\partial h} - 2h \dfrac{\partial I}{\partial b} + f \dfrac{\partial I}{\partial g} - g \dfrac{\partial I}{\partial f} = 0\,, & (24) \\[2ex]
a \dfrac{\partial I}{\partial a} + h \dfrac{\partial I}{\partial h} + b \dfrac{\partial I}{\partial b} + g \dfrac{\partial I}{\partial g} + f \dfrac{\partial I}{\partial f} + c \dfrac{\partial I}{\partial c} = 0\,. & (6)
\end{cases}
$$

This complete system, M_2, has six variables and four equations, and hence at least two independent solutions. In order to determine whether there exist other solutions we form the matrix of the coefficients of $\dfrac{\partial I}{\partial a}, \dfrac{\partial I}{\partial h}, \ldots,$

$$
\left\| \begin{matrix}
0 & 0 & 0 & a & h & 2g \\
0 & 0 & 0 & h & b & 2f \\
2h & -2h & b - a & f & -g & 0 \\
a & h & b & g & f & c
\end{matrix} \right\|
$$

That the number of independent solutions of M_2 be greater than two it is necessary that all fourth order determinants of this matrix vanish identically. But this is easily seen not to be the case. Take, for example, the determinant formed by the first four columns on the right, in which c appears but once. The coefficient of c, namely the minor of c, is not zero, and hence, since the $a, h, b, \ldots,$ are independent, the determinant considered cannot vanish identically. Therefore the system M_2 has but two independent solutions.

Hence a conic possesses only two independent invariants by the group of motions. The integration of M_2 will now give these invariants.

For convenience put, as usual,

$$
\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},
$$

$$
A \equiv bc = f^2,\ B \equiv ca - g^2,\ C \equiv ab - h^2,\ F \equiv gh - af,\ G = hf - bg,\ H \equiv fg - ch\,.
$$

Note the following consequences of these identities:

$$\left.\begin{aligned} &aA + hH + gG = hH + bB + fF = \ldots = \Delta, \\ &hA + bH + fg = gA + fH + cG = \ldots = 0, \\ &BC - F^2 = a\Delta,\ CA - G^2 = b\Delta, \ldots . \end{aligned}\right\} \qquad (26)$$

The equation (22) is equivalent to the following simultaneous system:

$$\frac{dg}{a} = \frac{df}{h} = \frac{dc}{2g};$$

whence at once are written the two integrals

$$B, \quad F.$$

Introducing these in equation (23) there results the partial differential equation

$$2F\frac{\partial I}{\partial B} - C\frac{\partial I}{\partial F} = 0$$

whose equivalent simultaneous system

$$\frac{dB}{2F} = -\frac{dF}{C}$$

gives the integral

$$BC - F^2,$$

or by the relations (26) simply

$$\Delta.$$

Substituting this as a variable in equation (24) and observing that the relations (26) render the coefficient of $\frac{\partial I}{\partial \Delta}$ zero, we have the partial differential equation:

$$2h\left[\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b}\right] - (a - b)\frac{\partial I}{\partial h} = 0.$$

The equivalent simultaneous system,

$$-\frac{da}{2h} = \frac{db}{2h} = \frac{dh}{a - b},$$

leaves the integrals

$$a + b, \quad C.$$

We have then the three solutions of (22), (23) and (24), viz.,

$$a + b, \quad C, \quad \Delta.$$

Equation (6), however, demands that the common solutions of the system M_2 be homogeneous and of the zero order of homogeneity.

The forms

$$I_1 \equiv \frac{(a+b)\Delta}{C^2}, \quad I_2 \equiv \frac{\Delta^2}{C^3} \tag{27}$$

are easily seen to satisfy these conditions and the system. Therefore

$$\frac{(a+b)\Delta}{C^2} \text{ and } \frac{\Delta^2}{C^3}$$

are two independent invariants of the conic $U = 0$ by the group of motions, all other invariants of $U = 0$ by the group of motions are functions of these two.

Since I_1 and I_2 are absolute invariants it is easy to give a geometrical interpretation to them, and thereby we arrive at a simple equation for determining the magnitude of the axes of a conic in terms of the parameters of its general equation. For, taking the equation of the conic in the normal form $V = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = 0$, we have

$$a = \frac{1}{\alpha^2}, \; b = \frac{1}{\beta^2}, \; c = -1, \; f = g = h = 0, \; \Delta = -\frac{1}{\alpha^2\beta^2}, \; C = +\frac{1}{\alpha^2\beta^2},$$

$$a + b = \frac{1}{\alpha^2} + \frac{1}{\beta^2};$$

hence

$$I_1 = -(\alpha^2 + \beta^2), \quad I_2 = +\alpha^2\beta^2.$$

Accordingly the axes of the conic $V = 0$ are invariant by the group of motions, and the two invariants I_1 and I_2 express respectively the sum of the squares and the product of the squares of the semi-axes of $V = 0$. Consequently, since the expressions I_1 and I_2 are absolute invariants, they are the expressions, respectively, for the sum of the squares and the product of the squares of the semi-axes of the conic $U = 0$; and the axes of $U = 0$ are invariant by the group of motions.

Hence the squares of the semi-axes of the conic $U = 0$ are given by the quadratic equation

$$\lambda^2 + I_1\lambda + I_2 = 0,$$

or

$$C^3\lambda^2 + (a+b)\,C\Delta\lambda + \Delta^2 = 0.\text{*} \tag{28}$$

A discussion of the roots of this equation would lead to an exhaustive classi-

* A similar but more complicated form is given without proof in the sixth edition of Salmon's Conic Sections, Miscellaneous Notes, p. 392.

fication of the curves of the second degree. Also the invariance of I_1 and I_2 constitutes the Congruence Criterium* of the curves of the second degree.

2. To determine the invariants of $U = 0$ by the group of similitude it is necessary to integrate the complete system :

$$S_2 \begin{cases} a\frac{\partial I}{\partial g} + h\frac{\partial I}{\partial f} + 2g\frac{\partial I}{\partial c} = 0, & (22) \\ h\frac{\partial I}{\partial g} + b\frac{\partial I}{\partial f} + 2f\frac{\partial I}{\partial c} = 0, & (23) \\ 2h\frac{\partial I}{\partial a} - (a-b)\frac{\partial I}{\partial h} - 2h\frac{\partial I}{\partial b} + f\frac{\partial I}{\partial g} - g\frac{\partial I}{\partial f} = 0, & (24) \\ 2a\frac{\partial I}{\partial a} + 2h\frac{\partial I}{\partial h} + 2b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} = 0, & (25) \\ a\frac{\partial I}{\partial a} + h\frac{\partial I}{\partial h} + b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} + c\frac{\partial I}{\partial c} = 0. & (6) \end{cases}$$

Here are five equations and six variables ; accordingly, the system has at least one solution. To determine if there are others it is only necessary to form the matrix as in the preceding problem, when it will be seen that all fifth order determinants of the matrix do not vanish identically ; consequently there are no additional solutions and the conic $U = 0$ has but one invariant by the group of similitude. This invariant may now be found by the integration of the system S_2, which is accomplished without difficulty when we avail ourselves of the integrals found in the preceding problem. The solution then is readily seen to be

$$I_3 \equiv \frac{(a+b)^2}{C}; \tag{29}$$

and the geometrical interpretation of I_3 is easily found to be

$$I_3 \equiv \frac{(a+b)^2}{C} = \frac{I_1^2}{I_2} = (\alpha^2 + \beta^2)\left[\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right], \dagger$$

where α and β are the semi-axes of $U = 0$. Hence the ratio of the axes of the conic is invariant by the group of similitude. The invariance of I_3 then includes the whole theory of the similar curves of the second degree. As a corollary the eccentricity depends only on the coefficients of the terms of the second degree since $I_3 \equiv \dfrac{(a+b)^2}{ab - h^2}$.

* Vid. loc. cit., pp. 673 et seq.

† It is to be observed that I_1 and I_2 are not invariants in these equations.

II.

Consider now the corresponding problems in space. Let

$$T \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}xz + 2a_{12}xy + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0\,, \qquad (30)$$

be the equation of the conicoid. Let

$$J(a_{ij})$$

be an invariant function of the parameters of $T = 0$ by a projective group of infinitesimal transformations in ordinary space. T is homogeneous in these parameters, hence J is a homogeneous function and of the zero order ; accordingly Euler's theorem gives

$$\Sigma a_{ij} \frac{\partial J}{\partial a_{ij}} = 0\,. \qquad (a_{ij} = a_{ji},\ i, j = 1, 2, 3, 4) \quad (31)$$

Also the invariance of J gives

$$\delta J \equiv \Sigma \frac{\partial J}{\partial a_{ij}} \delta a_{ij} = 0\,. \qquad (a_{ij} = a_{ji},\ i, j = 1, 2, 3, 4) \quad (32)$$

The group of motions in space is a six-parameter projective group and consists of the infinitesimal transformations :

$$\boxed{p \quad q \quad r \quad yp - xq \quad zq - yr \quad xr - zp} \qquad (33)$$

The group of similitude in space, also projective, contains the seven infinitesimal transformations :

$$\boxed{p \quad q \quad r \quad yp - xq \quad zq - yr \quad xr - zp \quad xp - yq + zr} \qquad (34)$$

where $p \equiv \dfrac{\partial f}{\partial x}$, $q \equiv \dfrac{\partial f}{\partial y}$, $r \equiv \dfrac{\partial f}{\partial z}$.

Operating on $T = 0$ with the several infinitesimal transformations of these groups, we have

$$\begin{aligned} pT^* \equiv\ & \delta_1 a_{11}x^2 + \delta_1 a_{22}y^2 + \delta_1 a_{33}z^2 + 2\delta_1 a_{23}yz + 2\delta_1 a_{13}xz + 2\delta_1 a_{12}xy \\ & + 2(a_{11} + \delta_1 a_{14})\,x + 2(a_{12} + \delta_1 a_{24})\,y + 2(a_{13} + \delta_1 a_{34})\,z \\ & + \delta_1 a_{44} + 2a_{14} = 0\,; \end{aligned} \qquad (35)$$

* It is to be noted that the notation pT, ... is not rigorously exact and that it is used only for purposes of identification.

$$qT \equiv \delta_2 a_{11}x^2 + \delta_2 a_{22}y^2 + \delta_2 a_{33}z^2 + 2\delta_2 a_{23}yz + 2\delta_2 a_{13}xz + 2\delta_2 a_{12}xy + 2(a_{12} + \delta_2 a_{14})x + 2(a_{22} + \delta_2 a_{24})y + 2(a_{23} + \delta_2 a_{44})z + \delta_2 a_{44} + 2a_{24} = 0; \tag{36}$$

$$rT \equiv \delta_3 a_{11}x^2 + \delta_3 a_{22}y^2 + \delta_3 a_{33}z^2 + 2\delta_3 a_{23}yz + 2\delta_3 a_{13}xz + 2\delta_3 a_{12}xy + 2(a_{13} + \delta_3 a_{14})x + 2(a_{23} + \delta_3 a_{24})y + 2(a_{33} + \delta_3 a_{34})z + \delta_3 a_{44} + 2a_{34} = 0; \tag{37}$$

$$(yp - xq)\,T \equiv (\delta_4 a_{11} - 2a_{12})x^2 + (\delta_4 a_{22} + 2a_{12})y^2 + \delta_4 a_{33}z^2 + 2(\delta_4 a_{23} + a_{13})yz + 2(\delta_4 a_{13} - a_{23})xz + 2(\delta_4 a_{12} + a_{11} - a_{22})xy + 2(\delta_4 a_{14} - a_{24})x + 2(\delta_4 a_{24} + a_{14})y + 2\delta_4 a_{34}z + \delta_4 a_{44} = 0; \tag{38}$$

$$(zq - xr)\,T \equiv \delta_5 a_{11}x^2 + (\delta_5 a_{22} - 2a_{23})y^2 + (\delta_5 a_{33} + 2a_{23})z^2 + 2(\delta_5 a_{23} + a_{22} - a_{33})yz + 2(\delta_5 a_{13} + a_{23})xz + 2(\delta_5 a_{23} - a_{13})xy + 2\delta_5 a_{14}x + 2(\delta_5 a_{24} - a_{34})y + 2(\delta_5 a_{34} + a_{24})z + \delta_5 a_{44} = 0; \tag{39}$$

$$(zr - zp)\,T \equiv (\delta_6 a_{11} + 2a_{13})x^2 + \delta_6 a_{22}y^2 + (\delta_6 a_{33} - 2a_{13})z^2 + 2(\delta_6 a_{23} - a_{12})yz + 2(\delta_6 a_{13} + a_{33} - a_{11})xz + 2(\delta_6 a_{12} + a_{23})xy + 2(\delta_6 a_{14} + a_{34})x + 2\delta_6 a_{24}y + 2(\delta_6 a_{34} - a_{14})z + \delta_6 a_{44} = 0; \tag{40}$$

$$(xp + yq + zr)\,T \equiv (\delta_7 a_{11} + 2a_{11})x^2 + (\delta_7 a_{22} + 2a_{22})y^2 + (\delta_7 a_{33} + 2a_{33})z^2 + 2(\delta_7 a_{23} + 2a_{23})yz + 2(\delta_7 a_{13} + 2a_{13})xz + 2(\delta_7 a_{12} + 2a_{12})xy + 2(\delta_7 a_{14} + a_{14})x + 2(\delta_7 a_{24} + a_{24})y + 2(\delta_7 a_{34} + a_{34})z + \delta_7 a_{44} = 0. \tag{41}$$

The above equations exist by virtue of the equation $T = 0$; hence comparing them successively with $T = 0$, we have respectively:

$$\frac{\delta_1 a_{11}}{a_{11}} = \frac{\delta_1 a_{22}}{a_{22}} = \frac{\delta_1 a_{33}}{a_{33}} = \frac{\delta_1 a_{23}}{a_{23}} = \frac{\delta_1 a_{13}}{a_{13}} = \frac{\delta_1 a_{12}}{a_{12}} = \frac{\delta_1 a_{14} + a_{11}}{a_{14}} = \frac{\delta_1 a_{24} + a_{12}}{a_{24}} = \frac{\delta_1 a_{34} + a_{13}}{a_{34}} = \frac{\delta_1 a_{44} + 2a_{14}}{a_{44}} = \rho_1; \tag{42}$$

$$\frac{\delta_2 a_{11}}{a_{11}} = \frac{\delta_2 a_{22}}{a_{22}} = \frac{\delta_2 a_{33}}{a_{33}} = \frac{\delta_2 a_{23}}{a_{23}} = \frac{\delta_2 a_{13}}{a_{13}} = \frac{\delta_2 a_{12}}{a_{12}} = \frac{\delta_2 a_{14} + a_{12}}{a_{14}} = \frac{\delta_2 a_{24} + a_{22}}{a_{24}} = \frac{\delta_2 a_{34} + a_{23}}{a_{34}} = \frac{\delta_2 a_{44} + 2a_{24}}{a_{44}} = \rho_2; \tag{43}$$

$$\frac{\delta_3 a_{11}}{a_{11}} = \frac{\delta_3 a_{22}}{a_{22}} = \frac{\delta_3 a_{33}}{a_{33}} = \frac{\delta_3 a_{23}}{a_{23}} = \frac{\delta_3 a_{13}}{a_{13}} = \frac{\delta_3 a_{12}}{a_{12}} = \frac{\delta_3 a_{14} + a_{13}}{a_{14}} = \frac{\delta_3 a_{24} + a_{23}}{a_{24}} = \frac{\delta_3 a_{34} + a_{33}}{a_{34}} = \frac{\delta_3 a_{44} + 2a_{34}}{a_{44}} = \rho_3; \tag{44}$$

$$\frac{\delta_4 a_{11} - 2a_{12}}{a_{11}} = \frac{\delta_4 a_{22} + 2a_{12}}{a_{22}} = \frac{\delta_4 a_{33}}{a_{33}} = \frac{\delta_4 a_{23} + a_{13}}{a_{23}} = \frac{\delta_4 a_{13} - a_{23}}{a_{13}} = \frac{\delta_4 a_{12} + a_{11} - a_{22}}{a_{12}}$$
$$= \frac{\delta_4 a_{14} - a_{24}}{a_{14}} = \frac{\delta_4 a_{24} + a_{14}}{a_{24}} = \frac{\delta_4 a_{34}}{a_{34}} = \frac{\delta_4 a_{44}}{a_{44}} = \rho_4 ; \quad (45)$$

$$\frac{\delta_5 a_{11}}{a_{11}} = \frac{\delta_5 a_{22} - a_{23}}{a_{22}} = \frac{\delta_5 a_{33} + 2a_{23}}{a_{33}} = \frac{\delta_5 a_{23} + a_{22} - a_{33}}{a_{23}} = \frac{\delta_5 a_{13} + a_{12}}{a_{13}} = \frac{\delta_5 a_{12} - a_{13}}{a_{12}}$$
$$= \frac{\delta_5 a_{14}}{a_{14}} = \frac{\delta_5 a_{24} - a_{34}}{a_{24}} = \frac{\delta_5 a_{34} + a_{24}}{a_{34}} = \frac{\delta_5 a_{44}}{a_{44}} = \rho_5 ; \quad (46)$$

$$\frac{\delta_6 a_{11} + 2a_{13}}{a_{11}} = \frac{\delta_6 a_{22}}{a_{22}} = \frac{\delta_6 a_{33} - 2a_{13}}{a_{33}} = \frac{\delta_6 a_{23} - a_{12}}{a_{23}} = \frac{\delta_6 a_{13} + a_{33} - a_{11}}{a_{13}} = \frac{\delta_6 a_{12} + a_{23}}{a_{12}}$$
$$= \frac{\delta_6 a_{14} + a_{34}}{a_{14}} = \frac{\delta_6 a_{24}}{a_{24}} = \frac{\delta_6 a_{34} - a_{14}}{a_{34}} = \frac{\delta_6 a_{44}}{a_{44}} = \rho_6 ; \quad (47)$$

$$\frac{\delta_7 a_{11} + 2a_{11}}{a_{11}} = \frac{\delta_7 a_{22} + 2a_{22}}{a_{22}} = \frac{\delta_7 a_{33} + 2a_{33}}{a_{33}} = \frac{\delta_7 a_{23} + 2a_{23}}{a_{23}} = \frac{\delta_7 a_{13} + 2a_{13}}{a_{13}}$$
$$= \frac{\delta_7 a_{12} + 2a_{12}}{a_{12}} = \frac{\delta_7 a_{14} + a_{14}}{a_{14}} = \frac{\delta_7 a_{24} + a_{24}}{a_{24}} = \frac{\delta_7 a_{34} + a_{24}}{a_{34}} = \frac{\delta_7 a_{44}}{a_{44}} = \rho_7 . \quad (48)$$

The values of the $\delta_r a_{ij}$, now to be found by the solutions of these systems of equations, must satisfy the equation (32). By virtue of equation (31) the indeterminate multipliers $\rho_1, \ldots, \rho_7$ disappear after the substitution of the $\delta_r a_{ij}$ in equation (32). Hence the solution of equations (42) to (48) with regard to the $\delta_r a_{ij}$ and the successive substitution of the systems of values of $\delta_r a_{ij}$ in the equation (32) leaves the following homogeneous linear partial differential equations of the first order, respectively :

$$a_{11}\frac{\partial J}{\partial a_{14}} + a_{12}\frac{\partial J}{\partial a_{24}} + a_{13}\frac{\partial J}{\partial a_{34}} + 2a_{14}\frac{\partial J}{\partial a_{44}} = 0 , \quad (49)$$

$$a_{12}\frac{\partial J}{\partial a_{14}} + a_{22}\frac{\partial J}{\partial a_{24}} + a_{23}\frac{\partial J}{\partial a_{34}} + 2a_{24}\frac{\partial J}{\partial a_{44}} = 0 , \quad (50)$$

$$a_{13}\frac{\partial J}{\partial a_{14}} + a_{23}\frac{\partial J}{\partial a_{24}} + a_{33}\frac{\partial J}{\partial a_{34}} + 2a_{34}\frac{\partial J}{\partial a_{44}} = 0 , \quad (51)$$

$$2a_{12}\left[\frac{\partial J}{\partial a_{11}} - \frac{\partial J}{\partial a_{22}}\right]$$
$$- a_{13}\frac{\partial J}{\partial a_{13}} + a_{23}\frac{\partial J}{\partial a_{23}} - (a_{11} - a_{22})\frac{\partial J}{\partial a_{12}} + a_{24}\frac{\partial J}{\partial a_{14}} - a_{14}\frac{\partial J}{\partial a_{24}} = 0 , \quad (52)$$

$$2a_{23}\left[\frac{\partial J}{\partial a_{22}} - \frac{\partial J}{\partial a_{33}}\right]$$
$$- (a_{22} - a_{33})\frac{\partial J}{\partial a_{23}} - a_{12}\frac{\partial J}{\partial a_{13}} + a_{13}\frac{\partial J}{\partial a_{12}} + a_{34}\frac{\partial J}{\partial a_{24}} - a_{24}\frac{\partial J}{\partial a_{34}} = 0 , \quad (53)$$

$$2a_{13}\left[\frac{\partial J}{\partial a_{33}}-\frac{\partial J}{\partial a_{11}}\right]$$

$$+\,a_{12}\frac{\partial J}{\partial a_{23}}-(a_{33}-a_{11})-a_{23}\frac{\partial J}{\partial a_{12}}-a_{34}\frac{\partial J}{\partial a_{14}}+a_{14}\frac{\partial J}{\partial a_{34}}=0\,, \quad (54)$$

$$2\left[a_{11}\frac{\partial J}{\partial a_{11}}+a_{22}\frac{\partial J}{\partial a_{22}}+a_{33}\frac{\partial J}{\partial a_{33}}+a_{23}\frac{\partial J}{\partial a_{23}}+a_{13}\frac{\partial J}{\partial a_{13}}+a_{12}\frac{\partial J}{\partial a_{12}}\right]$$

$$+\,a_{14}\frac{\partial J}{\partial a_{14}}+a_{24}\frac{\partial J}{\partial a_{24}}+a_{34}\frac{\partial J}{\partial a_{34}}=0\,. \quad (55)$$

3. Then to determine the invariants of the conicoid $T=0$ by the group of motions in space we have the complete system, M_3, of homogeneous linear partial differential equations of the first order composed of the equations

(49) (50) (51) (52) (53) (54) (31).

This is a system of seven equations in ten variables and hence has at least three solutions. By forming the matrix whose elements are the coefficients of the $\frac{\partial J}{\partial a_{ij}}$ in the above system it is readily seen that all seventh order determinants of the matrix do not vanish identically, and that accordingly there are no more than three independent solutions. In this matrix, namely,

$$\begin{Vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 2a_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{12} & a_{22} & a_{23} & 2a_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{13} & a_{23} & a_{33} & 2a_{34} \\
2a_{12} & -2a_{12} & 0 & -a_{13} & a_{23} & -(a_{11}-a_{22}) & a_{24} & -a_{14} & 0 & 0 \\
0 & 2a_{23} & -2a_{23} & -(a_{22}-a_{33}) & -a_{12} & a_{13} & 0 & a_{34} & -a_{24} & 0 \\
-2a_{13} & 0 & 2a_{13} & a_{12} & -(a_{33}-a_{11}) & -a_{23} & -a_{34} & 0 & a_{14} & 0 \\
a_{11} & a_{22} & a_{33} & a_{23} & a_{13} & a_{12} & a_{14} & a_{24} & a_{34} & a_{44}
\end{Vmatrix}$$

the term a_{44} occurs but once. Its coefficient in the determinant formed by the seven columns on the right is not zero; hence this determinant itself cannot be equal to zero, since the a_{ij} are independent of one another.

Hence the conicoid $T=0$ has but three independent invariants by the group of motions. These may now be found by integrating the complete system M_3.

For convenience put

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix},$$

and A_{ii}, $2A_{ij}$ identically equal respectively to the minors of D corresponding to a_i^i, a_{ij}.

Then proceeding with the system M_3 in exactly the same manner in which the system M_2 of the corresponding problem in the plane was dealt with, we find that the first three equations (49), (50) and (51) have the solution

$$A_{44}.$$

Further that the equations (52), (53) and (54) have in addition to this the solutions

$$a_{11} + a_{22} + a_{33}, \; a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{23}^2 - a_{13}^2 - a_{12}^2, \; D.$$

Finally the condition of homogeneity demanded by the equation (31) and the system are found to be satisfied by

$$J_1 \equiv \frac{DP}{A_{44}^2}, \quad J_2 \equiv \frac{D^2S}{A_{44}^3}, \quad J_3 \equiv \frac{D^3}{A_{44}^4},$$

where $S \equiv a_{11} + a_{22} + a_{33}$, and $P \equiv a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{23}^2 - a_{13}^2 - a_{12}^2$.

Analogous to the geometrical interpretation in the plane we find on comparison with the conicoid

$$W \equiv \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} - 1 = 0,$$

that

$$J_1 = -(\alpha^2 + \beta^2 + \gamma^2), \quad J_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2, \quad J_3 = -\alpha^2\beta^2\gamma^2.$$

Consequently the axes of $T = 0$ are invariants by the group of motions, and the squares of the semi-axes are given by the cubic equation

$$\mu^3 + J_1\mu^2 + J_2\mu + J_3 = 0$$

or

$$A_{44}^4\mu^3 + A_{44}^2DP\mu^2 + A_{44}D^2S\mu + D^3 = 0. \tag{56}$$

The surfaces of the second degree may be classified by a discussion of the roots of this equation. It will also be observed that the Congruence Criterium of surfaces of the second degree is expressed by the invariance of J_1, J_2, and J_3.

4. To find the invariants of the conicoid $T = 0$ by the group of similitude in space (34), we have only to add the partial differential equation (55) to the system M_3. The new system S_3 consisting of the partial differential equations

$$(49)\ (50)\ (51)\ (52)\ (53)\ (54)\ (55)\ (31)$$

is a complete system of eight equations in ten variables. As in the preceding problem, a_{44} appears but once in the matrix, whereby it is easily seen that all eighth order determinants of the matrix do not vanish identically. The direct integration is simple when the solutions already found, namely, J_1, J_2, J_3 above, are made use of, and the two independent solutions of the system appear in the forms

$$J_4 \equiv \frac{P^3}{A_{44}{}^2} = \frac{J_1^3}{J_3}; \quad J_5 \equiv \frac{PS}{A_{44}} = \frac{J_1 J_2}{J_3}* \tag{57}$$

As in the plane problem these invariants determine the ratios of the axes of the conicoid. Hence the ratios of the axes of the conicoid $T = 0$ are invariant by the group of similitude in space.† The invariance of J_4 and J_5 expresses the condition that surfaces of the second degree be similar.

5. The method of this note allows of extension directly to curves whose equations take the form

$$\frac{x^{2n}}{\alpha^{2n}} + \frac{y^{2n}}{\beta^{2n}} = 1\,,$$

to the surfaces

$$\frac{x^{2n}}{\alpha^{2n}} + \frac{y^{2n}}{\beta^{2n}} + \frac{z^{2n}}{\gamma^{2n}} = 1\,,$$

and to the corresponding surfaces in spaces of higher dimensions, since these surfaces possess axes of symmetry at right angles. Obviously, as already remarked the method is capable of general application to curves and surfaces of any order, but the geometrical interpretation is not always reached with the same facility as in the cases above. Moreover a complete theory of the invariants 1° of a conic and 2° of systems of conics by the projective group in the plane may be developed by determining these invariants by the projective subgroups of one, two, three, and four parameters in the first case, and of parameters one, two, and so on to one less in number than the number of different constants entering the system in the second case. A corresponding theory for surfaces of the second degree is not without interest, but much more complicated. It is hoped to present some of these results in a subsequent note.

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* The reader will observe that J_1, J_2, and J_3 are not invariants in these expressions.

† It is interesting to remark that the ratios of the axes of $T = 0$ are integrals of the complete system of partial differential equations S_3, and that the axes themselves are integrals of the system M_3. A similar remark applies to the corresponding problems in the plane.

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